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Weak ψ–ω Formulation for Unsteady Flows in 2D Multiply Connected Domains

M. Biava, D. Modugno, L. Quartapelle, and M. Stoppelli

Dipartimento di Ingegneria Aerospaziale, Politecnico di Milano, Via La Masa 34, 20158 Milan, Italy
E-mail: quartapelle@aero.polimi.it

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This paper describes a variational formulation for solving the time-dependent Navier–Stokes equations expressed in terms of the stream function and vorticity around multiple airfoils. This approach is an extension to the case of multiply connected domains of the weak formulation based on explicit viscous diffusion recently proposed by Guermond and Quartapelle. In their method the momentum equation was interpreted as a dynamical equation governing the evolution of the (weak) Laplacian of the stream function, while the Poisson equation for the latter was used as an expression to evaluate the distribution of the vorticity. Time discretizations with the viscous term made explicit were used, leading to the viscosity being split from the incompressibility, similarly to the primitive variable fractional-step method.

In the present work the multiconnectedness is addressed by introducing an influence matrix to determine the constant values of the stream function on the airfoils in a noniterative fashion. The explicit treatment of the viscous term leads to an influence matrix rooted in the harmonic problem instead of in the biharmonic problem occurring in methods enforcing integral conditions on the vorticity, such as the Glowinski–Pironneau method. The influence matrix changes at each time step or is constant depending on whether a semi-implicit or fully explicit treatment is adopted for the nonlinear term. The resulting split method is implemented using a first-order Euler backward difference or a second-order BDF scheme and linear finite elements. Numerical results are given and compared with the solutions obtained by means of the biharmonic formulation for multiply connected domains. © 2002 Elsevier Science (USA)

Key Words: incompressible Navier–Stokes equations; unsteady 2D flows; multiply connected domains; vorticity and stream function formulation; explicit viscous diffusion; influence matrix.

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CONTENTS

1. INTRODUCTION

The formulation of the incompressible Navier–Stokes equations for the stream function and vorticity has the advantage over the primitive variable approach of reducing the number of unknowns and of eliminating the incompressibility constraint, which can be difficult to satisfy numerically. On the other hand, for problems in 2D multiply connected domain solutions calculated by means of nonprimitive variables can represent physically admissible flows only provided that the corresponding pressure field is a single-valued function; cf., e.g., [1, 2]. This condition poses a technical difficulty and several methods have been proposed to overcome it.

Typically, in these methods additional unknowns are introduced, to represent the constant values of the stream function on the surface of the bodies immersed in the stream. This implies the introduction of additional conditions for the problem to have a unique solution. In the context of finite differences, these conditions are often obtained by expressing the line integral of the tangential projection of the momentum equation onto the surface of the bodies. See, for example, the work of Stella and Guj [14], for the steady problem, and the works of Daube [3] and of Shen and Loc [13], for the nonstationary problem.

In the framework of the Galerkin finite element formulation, Glowinski and Pironneau [6] have introduced a method for the biharmonic problem based on the uncoupled solution of the vorticity and stream function equations, where the constant values of the stream function on the immersed bodies are determined by means of optimal control theory. This leads to a small system of linear equations of order equal to the degree of connectedness of the domain. A similar uncoupled solution method also based on the biharmonic formulation has been developed in [8].

For problems in simply connected domains, recently Guermond and Quartapelle [9] introduced a variational formulation for the time-dependent vorticity and stream function equations with an *explicit* account of the viscous diffusion term. This method overcomes the complexity inherent in the fourth-order biharmonic formulation by expressing the governing equations directly as a system of uncoupled equations at the cost of reducing the numerical stability, which becomes only conditional, in comparison to biharmonic approaches enforcing integral conditions on the vorticity, such as the aforementioned Glowinski–Pironneau method.

The aim of the present paper is to extend the method with an explicit viscous diffusion described in [9] to the case of multiply connected domains. We derive a semi-discrete linearized problem by approximating the time derivative with finite differences and by dealing with the nonlinear term in either a semi-implicit or an explicit manner. At this level the stream function is expanded as the sum of a suitable set of functions leading to a small linear system to determine the additional stream function unknowns. The influence matrix associated with such a linear system has to be recalculated at each time step if the nonlinear term is treated in a semi-implicit manner; alternatively, it can be determined once and for all at the preprocessing stage if the nonlinear term is made explicit. A spatial discretization based on mixed finite elements is considered.

The content of the paper is organized as follows. Section 2 introduces the preliminary definitions and the functional spaces necessary to formulate the equations of incompressible plane flows, using both primitive and nonprimitive variables. In Section 3 we recall the classical and variational formulations of the Navier–Stokes problem, written in terms of velocity and pressure, and then we derive the equivalent system of equations for the vorticity and the stream function. A thorough analysis is made to show how the additional conditions associated with the multiple connectedness can be interpreted in the light of the equivalence of the pressure/velocity formulation with the vorticity/stream function formulation. The weak equations which guarantee the single valuedness of the pressure are obtained by means of a particular decomposition of the space of the test functions. Section 4 addresses the numerical approximation of the vorticity and stream function equations. We begin by considering a time discretization based on a first-order Euler scheme. Then a special decomposition of the stream function is introduced to decouple the governing equation from the relations accounting for the multiple connectedness of the domain. A spatial discretization based on a mixed finite element technique is described. Furthermore, we present a fully explicit implementation of the problem and introduce a second-order BDF time-stepping algorithm. In Section 5 a set of boundary conditions suitable for external flow problems is presented, including the general case of nonhomogeneous data. In Section 6 a numerical application of the proposed method using linear elements is presented. In particular, we consider the flow past a multibody airfoil at high incidence with massive separation. A comparison with the solutions obtained by means of the biharmonic formulation is also made. The last section is devoted to a few concluding remarks.

2. DEFINITIONS AND PRELIMINARIES

The preliminary definitions and the functional setting of the problem follow closely the treatment of [8] and are reported here for the sake of completeness.

Let Ω be an open bounded domain of $\mathbb{R}^2$ that is assumed to be connected but may be multiply connected and let $\Gamma = \partial\Omega$ be the boundary of Ω. We denote by Γ_0 the external boundary and by Γ_i, $1 \leq i \leq p$, the internal connected component of Γ, so that

$$\Gamma = \bigcup_{i=0}^{p} \Gamma_i.$$

It is assumed that each component Γ_i is Lipschitz continuous; in other words, Γ_i may have sharp but not cusped edges. This regularity condition is the minimal requirement if the Dirichlet data for the velocity on Γ_i are the trace of a uniform field.

In the following, the space of the real functions infinitely differentiable and of compact support in Ω is denoted by $\mathcal{D}(\Omega)$. The space of distributions on Ω is denoted by $\mathcal{D}'(\Omega)$. Spaces of vector-valued functions are hereafter denoted with boldface type, although no distinction is made in the notation of inner products and norms. As usual, $L^2(\Omega)$ denotes the space of real-valued square-integrable functions. We denote the inner product in $L^2(\Omega)$

by $(\cdot\,,\cdot)$ and let $\|\cdot\|_0$ be the corresponding norm. $H^m(\Omega)$, $m \geq 0$, is the set of distributions whose successive derivatives, up to order m, are square-integrable.

To have a unitary framework for the curl operator in a two-dimensional space we introduce

$$\nabla \times \cdots : \mathcal{D}'(\Omega) \longrightarrow \boldsymbol{\mathcal{D}}'(\Omega); \quad \phi \longmapsto \left(\frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial x}\right),$$

$$\nabla \times \cdots : \boldsymbol{\mathcal{D}}'(\Omega) \longrightarrow \mathcal{D}'(\Omega); \quad \boldsymbol{v} \longmapsto \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}.$$

$(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ is a unit base of $\mathbb{R}^2$ and $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ is a right-handed unit base of $\mathbb{R}^3$. Note that we have $\nabla \times \phi = \nabla\phi \times \hat{\boldsymbol{z}}$ and $\nabla \times \boldsymbol{v} = \hat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{v}$. The following identities will be useful later:

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega), \quad \nabla \times (\nabla \times \boldsymbol{v}) = -\nabla^2 \boldsymbol{v} + \nabla(\nabla \cdot \boldsymbol{v}),$$
$$\forall \phi \in \mathcal{D}'(\Omega), \quad \nabla \times (\nabla \times \phi) = -\nabla^2 \phi.$$

If nonhomogeneous boundary conditions are involved and provided that ϕ and $\boldsymbol{f}$ are smooth enough, we have the formula

$$(\nabla \times \boldsymbol{f}, \phi) = (\boldsymbol{f}, \nabla \times \phi) + \oint_\Gamma \boldsymbol{f} \cdot \boldsymbol{\tau}\, \phi,$$

where $\boldsymbol{\tau}$ is the oriented unit tangent of Γ such that $(\boldsymbol{n}, \boldsymbol{\tau}, \hat{\boldsymbol{z}})$ is a right-handed triad of unit vectors, $\boldsymbol{n}$ being the outward normal.

The analysis of the Navier–Stokes equations leads to the consideration of solenoidal velocity fields; hence we define

$$\boldsymbol{\mathcal{J}}(\Omega) \equiv \left\{ \boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega) \;\middle|\; \nabla \cdot \boldsymbol{v} = 0 \right\}$$

and we denote $\boldsymbol{J}_0^m(\Omega)$, $m \geq 0$, the completion of $\boldsymbol{\mathcal{J}}(\Omega)$ in $\boldsymbol{H}^m(\Omega)$. If Ω is open, bounded, and Lipschitz, then the spaces $\boldsymbol{J}_0^0(\Omega)$ and $\boldsymbol{J}_0^1(\Omega)$ are characterized by

$$\boldsymbol{J}_0^0(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \;\middle|\; \nabla \cdot \boldsymbol{v} = 0,\ \boldsymbol{n} \cdot \boldsymbol{v}_{|\Gamma} = 0 \right\},$$
$$\boldsymbol{J}_0^1(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \;\middle|\; \nabla \cdot \boldsymbol{v} = 0,\ \boldsymbol{v}_{|\Gamma} = 0 \right\}.$$

For the ψ–ω formulation of the 2D Navier–Stokes equations in a multiply connected region we need also to introduce the spaces

$$\Phi \equiv \left\{ \varphi \in H^1(\Omega) \;\middle|\; \varphi_{|\Gamma_0} = 0,\ \varphi_{|\Gamma_i} = C_i,\ \forall C_i \in \mathbb{R},\ 1 \leq i \leq p \right\},$$
$$\Psi \equiv \left\{ \psi \in H^2(\Omega) \;\middle|\; \psi_{|\Gamma_0} = 0,\ \psi_{|\Gamma_i} = C_i,\ \forall C_i \in \mathbb{R},\ 1 \leq i \leq p,\ \frac{\partial \psi}{\partial n}_{|\Gamma} = 0 \right\}.$$

Spaces Φ and Ψ, equipped with the norm of $H^1(\Omega)$ and $H^2(\Omega)$, respectively, are Hilbert spaces.

3. THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS

3.1. The Formulation in Natural Variables

Let us consider the time-dependent Navier–Stokes problem expressed in terms of velocity $\boldsymbol{u}$ and pressure p (per unit density). The strong form of the problem is the following:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\nabla^2\boldsymbol{u} + \nabla p = \boldsymbol{f}, & \text{in } (0,T)\times\Omega, \\ \nabla\cdot\boldsymbol{u} = 0, & \text{in } (0,T)\times\Omega, \\ \boldsymbol{u} = \boldsymbol{b}, & \text{in } (0,T)\times\Gamma, \\ \boldsymbol{u}_{|t=0} = \boldsymbol{u}_0, & \text{in } \Omega. \end{cases} \tag{3.1}$$

The velocity $\boldsymbol{b}$ specified on the boundary must satisfy the global condition

$$\oint_\Gamma \boldsymbol{n}\cdot\boldsymbol{b} = 0, \quad \text{for all } t > 0,$$

which follows from integrating the continuity equation $\nabla\cdot\boldsymbol{u} = 0$ over the domain Ω. On the other hand, the initial velocity field $\boldsymbol{u}_0$ is assumed to be solenoidal; i.e.,

$$\nabla\cdot\boldsymbol{u}_0 = 0.$$

Finally, the boundary and initial data $\boldsymbol{b}$ and $\boldsymbol{u}_0$ are assumed to satisfy the compatibility condition (see [17] or [11])

$$\boldsymbol{n}\cdot\boldsymbol{b}(s,0) = \boldsymbol{n}\cdot\boldsymbol{u}_{0|\Gamma}.$$

We now introduce the weak formulation of the problem. For the sake of simplicity, we consider first homogeneous conditions for the velocity on the entire boundary, namely, $\boldsymbol{b} = 0$. Boundary conditions for external flows with possibly nonhomogeneous data will be examined in Section 5. Given $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ (a body force) and $\boldsymbol{u}_0 \in \boldsymbol{J}_0^0(\Omega)$, we have the well-posed Navier–Stokes problem (cf., e.g., Lions [10, p. 69])

$$\begin{cases} \text{Find } \boldsymbol{u} \in L^2\big(0,T;\boldsymbol{J}_0^1(\Omega)\big) \cap C\big(0,T;\boldsymbol{J}_0^0(\Omega)\big) \quad \text{with } \boldsymbol{u}_{|t=0} = \boldsymbol{u}_0 \text{ such that} \\ \forall\boldsymbol{v} \in \boldsymbol{J}_0^1(\Omega), \quad \left(\dfrac{\partial\boldsymbol{u}}{\partial t}, \boldsymbol{v}\right) + a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}), \end{cases} \tag{3.2}$$

where a denotes the bilinear form

$$a(\boldsymbol{u},\boldsymbol{v}) = \nu\left(\nabla\boldsymbol{u}, \nabla\boldsymbol{v}\right)$$

whereas b is the trilinear form defined by

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \big((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}, \boldsymbol{w}\big).$$

Remark. Note that the variational formulation introduced above does not involve the pressure. This variable appears only if the space of the test functions is enlarged from $\boldsymbol{J}_0^1(\Omega)$ to $\boldsymbol{H}_0^1(\Omega)$. It can be shown that, if $\boldsymbol{u}$ satisfies (3.2), then there exists $p \in L^2(0, T; L^2(\Omega)/\mathbb{R})$, such that

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \quad \left(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v}\right) + a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v}) = -(\nabla p, \boldsymbol{v}).$$

3.2. The ψ–ω Formulation

By virtue of the well-known isomorphisms (see Girault and Raviart [15])

$$\nabla \times \cdots : \Psi \longrightarrow \boldsymbol{J}_0^1(\Omega),$$
$$\nabla \times \cdots : \Phi \longrightarrow \boldsymbol{J}_0^0(\Omega),$$

it is possible to replace the test functions of $\boldsymbol{J}_0^1(\Omega)$ by those of $\nabla \Psi \times \hat{z}$ in (3.2) to obtain the problem

$$\begin{cases} \text{Find } \psi \in L^2(0, T; \Psi) \cap C(0, T; \Phi) \text{ and} \\ \omega \in L^2(0, T; L^2(\Omega)) \cap C(0, T; H^{-1}(\Omega)) \text{ such that} \\ \forall \varphi \in \Phi, \quad \big(\nabla(\psi)_{t=0}, \nabla\varphi\big) = \big(\boldsymbol{u}_0, \nabla\varphi \times \hat{z}\big), \quad \text{and for all } t > 0 \\ \forall \psi' \in \Psi, \quad \left(\dfrac{\partial \nabla\psi}{\partial t}, \nabla\psi'\right) - \nu\left(\omega, \nabla^2\psi'\right) + \big(J(\omega, \psi), \psi'\big) = \big(\boldsymbol{f}, \nabla\psi' \times \hat{z}\big), \\ \forall v \in L^2(\Omega), \quad \big(\omega, v\big) = -\big(\nabla^2\psi, v\big), \end{cases} \tag{3.3}$$

where $J(\omega, \psi)$ denotes the Jacobian determinant, ψ is the stream function, and ω is the vorticity, defined by

$$\begin{cases} \boldsymbol{u} = \nabla\psi \times \hat{z}, \\ \omega = \hat{z} \cdot \nabla \times \boldsymbol{u}. \end{cases}$$

The proof of the equivalence of problems (3.2) and (3.3) is given in [7].

3.3. Multiple Connectedness

To approach the multiple connectedness of the domain it is necessary to introduce additional vector spaces. First we define

$$\Psi_0 \equiv \left\{\psi \in \Psi \;\middle|\; \psi_{|\cup_{j=1}^p \Gamma_j} = 0\right\} = H_0^2(\Omega).$$

Then we assume that we have at hand p functions $k_1, \ldots, k_p$ such that

$$\forall i, j = 1, \ldots, p, \qquad k_i \in \Psi, \qquad k_{i|\Gamma_j} = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol. We denote by ${}^{\Gamma}\mathbb{K}$ the space defined by

$${}^{\Gamma}\mathbb{K} \equiv \operatorname{span}\langle k_1, \ldots, k_p\rangle.$$

Since $k_1, \ldots, k_p$ are linearly independent, ${}^{\Gamma}\mathbb{K}$ is a finite-dimensional Hilbert space of dimension p. Clearly, ${}^{\Gamma}\mathbb{K}$ is nontrivial only if the domain is multiply connected.

By virtue of the above definitions, we have the following decomposition:

$$\Psi = \Psi_0 \oplus {}^{\Gamma}\mathbb{K}. \tag{3.4}$$

We are now in a position to interpret the problem (3.3) in terms of distributions, for which a mixed discrete approximation can be built. By restricting the test functions to Ψ_0 and then to ${}^{\Gamma}\mathbb{K}$, we obtain the following formulation:

$$\begin{cases} \text{Find } \psi \in L^2(0,T;\Psi) \cap C(0,T;\Phi) \text{ and} \\ \omega \in L^2(0,T;L^2(\Omega)) \cap C(0,T;H^{-1}(\Omega)) \text{ such that} \\ \forall \varphi \in \Phi, \quad \left(\nabla(\psi)_{t=0}, \nabla\varphi\right) = \left(\boldsymbol{u}_0, \nabla\varphi \times \hat{z}\right), \quad \text{and for all } t > 0 \\ \forall \psi' \in \Psi_0, \quad \left(\dfrac{\partial \nabla\psi}{\partial t}, \nabla\psi'\right) - \nu\left(\omega, \nabla^2\psi'\right) + \left(\omega\, \hat{z} \times \nabla\psi, \nabla\psi'\right) = \left(\boldsymbol{f}, \nabla\psi' \times \hat{z}\right), \\ \forall i = 1,\ldots,p, \quad \left(\dfrac{\partial \nabla\psi}{\partial t}, \nabla k_i\right) - \nu\left(\omega, \nabla^2 k_i\right) + \left(\omega\, \hat{z} \times \nabla\psi, \nabla k_i\right) = \left(\boldsymbol{f}, \nabla k_i \times \hat{z}\right), \\ \forall v \in L^2(\Omega), \quad (\omega, v) = -\left(\nabla^2\psi, v\right). \end{cases} \tag{3.5}$$

Remark. Observe that an equation in strong form governing the dynamics of the Laplacian of ψ could be obtained by "taking the curl" of the momentum equation. However, by doing so we lose the information, which is instead obtained by testing the momentum equation of problem (3.1) against the curl of the p functions k_i, $1 \le i \le p$.

4. NUMERICAL APPROXIMATIONS

4.1. Time Discretization

For simplicity of exposition we consider first an approximation of the time derivative by means of a first-order Euler scheme, taking into account the viscous term explicitly and the nonlinear term in a semi-implicit manner. This gives a semi-discrete linear problem, to which we restrict our attention initially.

We denote with $\bar{\psi}$ and $\bar{\omega}$ the stream function and the vorticity evaluated at the previous time level t_n, while ψ and ω denote the unknown functions at the current time t_{n+1}.

To solve the time-discretized problem for ψ as a standard Dirichelet boundary value problem for the Laplacian operator, we introduce the constant values Ξ_i, $1 \le i \le p$, assumed by ψ at t_{n+1} on each internal boundary. Physically, these constants are associated with the amount of fluid that flows between neighboring bodies. Denoting by $\boldsymbol{f}$ the body force $\boldsymbol{f}^{n+1}$, the semi-discrete problem reads

$$\begin{cases} \text{Find } \psi \in \Psi \quad \text{and} \quad \omega \in L^2(\Omega) \text{ such that} \\ \forall \psi' \in \Psi_0, \quad \gamma\left(\nabla\psi, \nabla\psi'\right) + \left(\bar{\omega}\, \hat{z} \times \nabla\psi, \nabla\psi'\right) = r(\psi'), \\ \forall i = 1,\ldots,p, \quad \gamma\left(\nabla\psi, \nabla k_i\right) + \left(\bar{\omega}\, \hat{z} \times \nabla\psi, \nabla k_i\right) = r(k_i), \\ \psi_{|\Gamma_i} = \Xi_i, \quad 1 \le i \le p, \end{cases} \tag{4.1}$$

where $\gamma = 1/\Delta t$, $\Xi_i \in \mathbb{R}$, $i = 1, \ldots, p$, and $r(\phi)$ is the linear form defined by

$$r(\phi) = \big(\nabla\big(\gamma\bar{\psi} - \nu\bar{\omega}\big), \nabla\phi\big) + \big(\boldsymbol{f}, \nabla\phi \times \hat{z}\big).$$

The equation for the vorticity field ω at time t_{n+1} is uncoupled and will be considered Later.

4.2. Decomposition Method

One of the main difficulties associated with the ψ–ω formulation (4.1) of the incompressible Navier–Stokes equations in multiply connected domains is that one has to determine the values of the additional unknowns Ξ_i, $1 \le i \le p$.

In the light of the consideration above, it is natural to set

$$\psi = \psi_0 + \sum_{i=1}^{p} \Xi_i\, \psi_i, \tag{4.2}$$

where $\psi_0 \in \Psi_0$ and $\psi_i \in \Psi$, $1 \le i \le p$. The functions ψ_i can be conveniently defined to decouple the problem of determining ψ_0 from that of determining the constants Ξ_i.

Indeed, let the functions ψ_i, $1 \le i \le p$, be the solution to the following variational problems:

$$\begin{cases} \text{Find } \psi_i \in \Psi \text{ such that} \\ \forall\psi' \in \Psi_0, \quad \gamma\,\big(\nabla\psi_i, \nabla\psi'\big) + \big(\bar{\omega}\,\hat{z} \times \nabla\psi_i, \nabla\psi'\big) = 0, \\ \psi_{i|\Gamma_j} = \delta_{ij}, \quad 1 \le j \le p. \end{cases} \tag{4.3}$$

Thus, when the expression (4.2) is substituted into the semi-discrete problem (4.1), the latter decouples and gives rise to a variational problem for ψ_0 and to a $p \times p$ linear system for the unknown constants Ξ_i. The function ψ_0 is in fact the solution to the problem

$$\begin{cases} \text{Find } \psi_0 \in \Psi_0 \text{ such that} \\ \forall\psi' \in \Psi_0, \quad \gamma\,\big(\nabla\psi_0, \nabla\psi'\big) + \big(\bar{\omega}\,\hat{z} \times \nabla\psi_0,\, \nabla\psi'\big) = r(\psi'). \end{cases} \tag{4.4}$$

On the other hand, the set of equations controlling the constants Ξ_i, $1 \le i \le p$, is given by the conditions in (4.1) testing the momentum equation against the curl of functions of ${}^{\Gamma}\mathbb{K}$. The resulting linear system is

$$\boldsymbol{A}^{[\bar{\omega}]}\,\boldsymbol{\Xi} = \boldsymbol{\beta}^{[\bar{\omega}]}, \tag{4.5}$$

where

$$A_{ij}^{[\bar{\omega}]} = \gamma\,\big(\nabla\psi_j, \nabla k_i\big) + \big(\bar{\omega}\,\hat{z} \times \nabla\psi_j, \nabla k_i\big),$$

$$\beta_i^{[\bar{\omega}]} = -\gamma\,\big(\nabla\psi_0, \nabla k_i\big) - \big(\bar{\omega}\,\hat{z} \times \nabla\psi_0, \nabla k_i\big) + r(k_i).$$

Once the stream function ψ has been obtained, the vorticity field ω at the new time level t_{n+1} is given by

$$\begin{cases} \text{Find } \omega \in L^2(\Omega) \text{ such that} \\ \forall v \in L^2(\Omega), \quad \big(\omega, v\big) = -\big(\nabla^2\psi, v\big). \end{cases} \tag{4.6}$$

Remark. The treatment of multiconnectedness has been translated into the definition of the influence matrix $\boldsymbol{A}^{[\bar{\omega}]}$ and into the solution to the associated small (nonsymmetric if $\bar{\omega} \neq 0$) linear system (4.5). This is the distinctive element of the proposed method.

4.3. Spatial Discretization

Thanks to the decoupling between the variational problem for ψ_0 and the system giving the constants Ξ_i, we are able to introduce a mixed finite element approximation to the problem based on standard techniques developed for simply connected domains.

Let $\mathcal{T}_h$ be a regular triangulation of Ω and let $\mathbb{P}_\ell$ be the space of polynomials of two variables of degree less than or equal to ℓ. We introduce the finite-dimensional Hilbert spaces

$$X_h \equiv \big\{\varphi_h \,\big|\, \varphi_h \in C^0(\Omega),\ \varphi_{h|T} \in \mathrm{P}_\ell,\ \forall T \in \mathcal{T}_h\big\},$$
$$\Psi_h \equiv \big\{\varphi_h \in X_h \,\big|\, \varphi_{h|\Gamma_0} = 0, \varphi_{h|\Gamma_i} = C_i, C_i \in \mathrm{R}, 1 \le i \le p\big\},$$

T being every triangle of $\mathcal{T}_h$. An *external* approximation to the space Ψ_0 can be built as follows:

$$\Psi_{0,h} \equiv \big\{\varphi_h \in \Psi_h \,\big|\, \varphi_{h|\cup_{i=1}^p \Gamma_i} = 0\big\}.$$

Then we build an external approximation to ${}^\Gamma\mathbb{K}$ by defining the functions k_i, $1 \le i \le p$, satisfying the conditions

$$\forall i, j = 1, \dots, p, \quad k_{i,h} \in \Psi_h, \quad k_{i,h|\Gamma_j} = \delta_{ij}.$$

If we set ${}^\Gamma\mathbb{K}_h \equiv \operatorname{span}\langle k_{1,h}, \dots, k_{p,h}\rangle$, we have the decomposition

$$\Psi_h = \Psi_{0,h} \oplus {}^\Gamma\mathbb{K}_h. \tag{4.7}$$

We are now in a position to state the formulation of the fully discrete problem. The spatially discrete counterpart of the decomposition (4.2) is

$$\psi_h = \psi_{0,h} + \sum_{i=1}^{p} \Xi_i\, \psi_{i,h}, \tag{4.8}$$

where the functions $\psi_{i,h}$, $1 \le i \le p$, are assumed to be the solutions of the following discrete problems:

$$\begin{cases} \text{Find } \psi_{i,h} \in \Psi_h \text{ such that} \\ \forall \phi_h \in \Psi_{0,h}, \quad \gamma\,\big(\nabla\psi_{i,h}, \nabla\phi_h\big) + \big(\bar{\omega}_h\, \hat{\boldsymbol{z}} \times \nabla\psi_{i,h}, \nabla\phi_h\big) = 0, \\ \psi_{i,h|\Gamma_j} = \delta_{ij}, \quad 1 \le j \le p. \end{cases} \tag{4.9}$$

It follows that the function $\psi_{0,h}$ can be obtained by solving the linear problem

$$\begin{cases} \text{Find } \psi_{0,h} \in \Psi_{0,h} \text{ such that} \\ \forall \phi_h \in \Psi_{0,h}, \quad \gamma\left(\nabla\psi_{0,h}, \nabla\phi_h\right) + \left(\bar{\omega}_h\, \hat{\boldsymbol{z}} \times \nabla\psi_{0,h}, \nabla\phi_h\right) = r_h(\phi_h), \end{cases} \tag{4.10}$$

where the linear form $r_h(\phi_h)$ is defined as

$$r_h(\phi_h) = \left(\nabla\left(\gamma\bar{\psi}_h - \nu\bar{\omega}_h\right), \nabla\phi_h\right) + \left(\boldsymbol{f}, \nabla\phi_h \times \hat{\boldsymbol{z}}\right).$$

The small (nonsymmetric) linear system giving the additional stream function unknowns Ξ_i, $1 \leq i \leq p$, now reads

$$\boldsymbol{A}_h^{[\bar{\omega}_h]}\, \boldsymbol{\Xi} = \boldsymbol{\beta}_h^{[\bar{\omega}_h]}, \tag{4.11}$$

where

$$A_{h,ij}^{[\bar{\omega}_h]} = \gamma\left(\nabla\psi_{j,h}, \nabla k_{i,h}\right) + \left(\bar{\omega}_h\, \hat{\boldsymbol{z}} \times \nabla\psi_{j,h}, \nabla k_{i,h}\right),$$

$$\beta_{h,i}^{[\bar{\omega}_h]} = -\gamma\left(\nabla\psi_{0,h}, \nabla k_{i,h}\right) - \left(\bar{\omega}_h\, \hat{\boldsymbol{z}} \times \nabla\psi_{0,h}, \nabla k_{i,h}\right) + r_h(k_{i,h}).$$

Equation (4.6) for the vorticity must be integrated by parts to be consistent with our mixed finite element approximation, i.e., piecewise polynomials and C^0 approximation. Therefore the vorticity field ω_h is given by the solution of the problem

$$\begin{cases} \text{Find } \omega_h \in X_h \text{ such that} \\ \forall v_h \in X_h, \quad \left(\omega_h, v_h\right) = \left(\nabla\psi_h, \nabla v_h\right). \end{cases} \tag{4.12}$$

Remark. There is a certain freedom in choosing the test functions $k_{i,h}$, $1 \leq i \leq p$, defined by the conditions in Section (4.3). For example, if linear finite elements are used, one can choose the p "belt functions" associated with the internal boundaries, i.e., $k_{i,h}$ is the function taking the value 1 on every boundary node belonging to triangles having at least one node on Γ_i, and 0 elsewhere.

4.4. The Fully Explicit Scheme

We now develop a fully explicit formulation of the problem. The modification consists simply in replacing ψ_h with $\bar{\psi}_h$ in the nonlinear term of the momentum equation in (4.1). The decomposition (4.8) remains formally the same, but the functions $\psi_{i,h}$, $1 \leq i \leq p$, now are the solution of the following Laplace problems, which can be solved once and for all at the preprocessing stage:

$$\begin{cases} \text{Find } \psi_{i,h} \in \Psi_h \text{ such that} \\ \forall \phi_h \in \Psi_{0,h}, \quad \gamma\left(\nabla\psi_{i,h}, \nabla\phi_h\right) = 0, \\ \psi_{i,h|\Gamma_j} = \delta_{ij}, \quad 1 \leq j \leq p. \end{cases} \tag{4.13}$$

The variational problem for $\psi_{0,h}$ now reads

$$\begin{cases} \text{Find } \psi_{0,h} \in \Psi_{0,h} \text{ such that} \\ \forall \phi_h \in \Psi_{0,h}, \quad \gamma\left(\nabla\psi_{0,h}, \nabla\phi_h\right) = r_h^{\text{expl}}(\phi_h), \end{cases} \tag{4.14}$$

with the linear form $r_h^{\text{expl}}(\phi_h)$ defined as

$$r_h^{\text{expl}}(\phi_h) = -\big(\bar{\omega}_h\ \hat{z} \times \nabla\bar{\psi}_h, \nabla\phi_h\big) + \big(\nabla(\gamma\bar{\psi}_h - \nu\bar{\omega}_h), \nabla\phi_h\big) + \big(\boldsymbol{f}, \nabla\phi_h \times \hat{z}\big).$$

The linear system associated with the multiple connectedness becomes

$$\boldsymbol{A}_h\ \boldsymbol{\Xi} = \boldsymbol{\beta}_h, \tag{4.15}$$

where now

$$\begin{aligned} A_{h,ij} &= \gamma\ \big(\nabla\psi_{j,h}, \nabla k_{i,h}\big),\\ \beta_{h,i} &= -\gamma\ \big(\nabla\psi_{0,h}, \nabla k_{i,h}\big) + r_h^{\text{expl}}(k_{i,h}). \end{aligned}$$

Observe that also the influence matrix $\boldsymbol{A}_h$ can be calculated once and for all during preprocessing since the functions $\psi_{i,h}$ are now time-independent.

The vorticity field ω_h is still obtained by solving problem (4.12).

4.5. The Second-Order BDF Scheme

The semi-implicit method can be modified to obtain high-order accuracy in time. For instance, this can be done by approximating the time derivative using a second-order BDF scheme combined with linear extrapolation of the vorticity in the nonlinear term.

Let ψ_h and ω_h be the unknowns at the current time step t_{n+1}, $\bar{\psi}_h$ and $\bar{\omega}_h$ the solution at the time step t_n and $\bar{\bar{\psi}}_h$ and $\bar{\bar{\omega}}_h$ the vorticity and the stream function evaluated at the older time step t_{n-1}. For convenience, we introduce an estimated vorticity field at time t_{n+1} by means of a linear extrapolation in time, namely,

$$\omega_h^\star = 2\,\bar{\omega}_h - \bar{\bar{\omega}}_h. \tag{4.16}$$

The scheme is initialized by evaluating (ψ_h^0, ω_h^0) and (ψ_h^1, ω_h^1). In particular, ψ_h^0 and ω_h^0 are determined from the initial data, whereas ψ_h^1 can be obtained by many means—for example, it can be calculated using a second-order Runge–Kutta method; from ψ_h^1 one evaluates ω_h^1 easily. Then, for each $n \geq 1$, we proceed as described hereafter.

First, ψ_h is decomposed as usual:

$$\psi_h = \psi_{0,h} + \sum_{i=1}^{p} \Xi_i\ \psi_{i,h}. \tag{4.17}$$

The functions $\psi_{i,h}$ and $\psi_{0,h}$ in (4.17) are obtained respectively as the solutions of the problems

$$\begin{cases} \text{Find } \psi_{i,h} \in \Psi_h \text{ such that}\\ \forall\phi_h \in \Psi_{0,h}, \quad \frac{3}{2}\gamma\ \big(\nabla\psi_{i,h}, \nabla\phi_h\big) + \big(\omega_h^\star\ \hat{z} \times \nabla\psi_{i,h}, \nabla\phi_h\big) = 0,\\ \psi_{i,h|\Gamma_j} = \delta_{ij}, \quad 1 \leq j \leq p, \end{cases} \tag{4.18}$$

and

$$
\begin{cases}\text{Find } \psi_{0,h} \in \Psi_{0,h} \text{ such that} \\ \forall \phi_h \in \Psi_{0,h}, \quad \frac{3}{2}\gamma \left(\nabla \psi_{0,h}, \nabla \phi_h\right) + \left(\omega_h^\star \, \hat{z} \times \nabla \psi_{0,h}, \nabla \phi_h\right) = r_h^{\mathrm{BDF}}(\phi_h),\end{cases} \tag{4.19}
$$

with the linear form $r_h^{\mathrm{BDF}}(\phi_h)$ given by

$$
r_h^{\mathrm{BDF}}(\phi_h) = \left(\gamma \nabla \left(2\, \bar{\psi}_h - \bar{\bar{\psi}}_h/2\right) - \nu \nabla \omega_h^\star, \nabla \phi_h\right) + \left(\boldsymbol{f}, \nabla \phi_h \times \hat{z}\right).
$$

Finally, the constants Ξ_i are calculated by solving the $p \times p$ linear system

$$
\boldsymbol{A}_h^{[\omega_h^\star]}\, \boldsymbol{\Xi} = \boldsymbol{\beta}_h^{[\omega_h^\star]}, \tag{4.20}
$$

where

$$
\begin{aligned}
A_{h,ij}^{[\omega_h^\star]} &= \tfrac{3}{2}\gamma \left(\nabla \psi_{j,h}, \nabla k_{i,h}\right) + \left(\omega_h^\star \, \hat{z} \times \nabla \psi_{j,h}, \nabla k_{i,h}\right), \\
\beta_{h,i}^{[\omega_h^\star]} &= -\tfrac{3}{2}\gamma \left(\nabla \psi_{0,h}, \nabla k_{i,h}\right) - \left(\omega_h^\star \, \hat{z} \times \nabla \psi_{0,h}, \nabla k_{i,h}\right) + r_h^{\mathrm{BDF}}(k_{i,h}).
\end{aligned}
$$

The vorticity field ω_h is evaluated by means of (4.12).

5. MORE GENERAL BOUNDARY CONDITIONS

5.1. Boundary Conditions for External Flows

To deal with external flow problems we are required to introduce more general boundary conditions. Here we rely upon the analysis found in Guermond and Quartapelle [8], which is repeated here for the sake of completeness.

Accordingly, it is assumed that the external boundary consists of three nonoverlapping parts,

$$
\Gamma_0 = \Gamma_{0,0} \cup \Gamma_{0,n} \cup \Gamma_{0,\tau},
$$

and we make the hypothesis that $\Gamma_{0,0} \cup \Gamma_{0,n}$ is connected. We enforce the velocity on $\Gamma_{0,0}$ (i.e., no-slip condition), the normal component of the velocity on $\Gamma_{0,n}$, and the tangential component on $\Gamma_{0,\tau}$.

It is natural to introduce the spaces

$$
\begin{aligned}
\boldsymbol{H} &\equiv \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \;\middle|\; \nabla\cdot \boldsymbol{v} = 0,\ \boldsymbol{n}\cdot \boldsymbol{v}_{|\Gamma_{0,0}\cup\Gamma_{0,n}\cup_{j=1}^{p}\Gamma_j} = 0 \right\}, \\
\boldsymbol{V} &\equiv \left\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \;\middle|\; \nabla\cdot \boldsymbol{v} = 0,\ \boldsymbol{v}_{|\Gamma_{0,0}\cup_{j=1}^{p}\Gamma_j} = 0,\ \boldsymbol{\tau}\cdot \boldsymbol{v}_{|\Gamma_{0,\tau}} = 0,\ \boldsymbol{n}\cdot \boldsymbol{v}_{|\Gamma_{0,n}} = 0 \right\}.
\end{aligned}
$$

Thus, given $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ and $\boldsymbol{u}_0 \in \boldsymbol{H}$, the problem (3.2) now reads

$$
\begin{cases}\text{Find } \boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap C(0,T;\boldsymbol{H}) \quad \text{with} \quad \boldsymbol{u}_{|t=0} = \boldsymbol{u}_0 \quad \text{such that} \\ \forall \boldsymbol{v} \in \boldsymbol{V}, \quad \left(\dfrac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v}\right) + a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = \left(\boldsymbol{f}, \boldsymbol{v}\right),\end{cases} \tag{5.1}
$$

where $a : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ is the continuous bilinear form defined by

$$a(\boldsymbol{u}, \boldsymbol{v}) = \nu\big(\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}\big),$$

and $b : \boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ is the continuous trilinear form defined as follows:

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \big((\nabla \times \boldsymbol{u}) \times \boldsymbol{v}, \boldsymbol{w}\big).$$

This choice of the form b leads us to the so-called rotational form of the Navier–Stokes problem and means that we want to work with the "total" pressure $p_{\mathrm{tot}} = p + |\boldsymbol{u}|^2/2$. Note that, even though the formulation above does not involve the pressure explicitly, it enforces a natural boundary condition on it:

$$p_{|\Gamma_{0,\tau}} = 0.$$

The ψ–ω formulation of the problem above under general boundary conditions requires the introduction of the following Hilbert spaces:

$$\Phi \equiv \big\{\varphi \in H^1(\Omega) \;\big|\; \varphi_{|\Gamma_{0,0}} = 0, \varphi_{|\Gamma_j} = C_j, C_j \in \mathrm{R}, 1 \le j \le p\big\},$$
$$\Psi \equiv \left\{\psi \in H^2(\Omega) \;\middle|\; \psi_{|\Gamma_{0,0}} = 0, \psi_{|\Gamma_j} = C_j, C_j \in \mathrm{R}, 1 \le j \le p, \frac{\partial \psi}{\partial n}_{|\Gamma \setminus \Gamma_{0,n}} = 0\right\}.$$

(For notational simplicity, the same symbols as before are retained.) As seen in Section 3.3, to deal with a multiply connected domain, we need to introduce the additional spaces

$$\Psi_0 \equiv \big\{\psi \in \Psi \;\big|\; \psi_{|\cup_{j=1}^{p}\Gamma_j} = 0\big\} \quad \text{and} \quad {}^{\Gamma}\mathbb{K} \equiv \operatorname{span}\langle k_1, \dots, k_p\rangle,$$

where the functions k_i are required to satisfy the conditions

$$\forall i, j = 1, \dots, p, \quad k_i \in \Psi, \quad k_{i|\Gamma_j} = \delta_{ij}.$$

Thanks to the isomorphisms (see Girault and Raviart [5])

$$\nabla \times \cdots : \Psi \to \boldsymbol{H},$$
$$\nabla \times \cdots : \Phi \to \boldsymbol{V},$$

and to the decomposition $\Psi = \Psi_0 \oplus {}^{\Gamma}\mathbb{K}$, the 2D Navier–Stokes problem formulated in velocity and pressure under general but homogeneous boundary conditions is equivalent to the following problem for the stream function and vorticity:

$$\begin{cases} \text{Find } \psi \in L^2(0, T; \Psi(\Omega)) \cap C(0, T; \Phi) \text{ and} \\ \omega \in L^2(0, T; L^2(\Omega)) \cap C(0, T; H^{-1}(\Omega)) \text{ such that} \\ \forall \varphi \in \Phi, \quad \big(\nabla(\psi)_{t=0}, \nabla\varphi\big) = \big(\boldsymbol{u}_0, \nabla\varphi \times \hat{z}\big), \quad \text{and for all } t > 0 \\ \forall \psi' \in \Psi_0, \quad \left(\dfrac{\partial \nabla\psi}{\partial t}, \nabla\psi'\right) - \nu\left(\omega, \nabla^2\psi'\right) - \left(\omega\, \hat{z} \times \nabla\psi, \nabla\psi'\right) = \left(\boldsymbol{f}, \nabla\psi' \times \hat{z}\right), \\ \forall i = 1, \dots, p, \quad \left(\dfrac{\partial \nabla\psi}{\partial t}, \nabla k_i\right) - \nu\left(\omega, \nabla^2 k_i\right) - \left(\omega\, \hat{z} \times \nabla\psi, \nabla k_i\right) = \left(\boldsymbol{f}, \nabla k_i \times \hat{z}\right), \\ \forall v \in L^2(\Omega), \quad (\omega, v) = -\big(\nabla^2\psi, v\big). \end{cases} \tag{5.2}$$

5.2. Nonhomogeneous Boundary Conditions

We now take into account nonhomogeneous boundary conditions. We enforce the *essential* boundary conditions on the velocity

$$\boldsymbol{u}_{|\Gamma_{0,0}\cup_{i=1}^{p}\Gamma_i} = \boldsymbol{u}_\Gamma(s,t), \quad \boldsymbol{n}\cdot\boldsymbol{u}_{|\Gamma_{0,n}} = u_n(s,t), \quad \boldsymbol{\tau}\cdot\boldsymbol{u}_{|\Gamma_{0,\tau}} = u_\tau(s,t),$$

where s is the arclength parameter for Γ. On the other hand, we enforce on the vorticity and pressure the *natural* boundary conditions

$$\omega_{|\Gamma_{0,n}} = c(s,t) \quad \text{and} \quad p_{|\Gamma_{0,\tau}} = q(s,t).$$

Let us define the following boundary data for ψ and its normal derivative:

$$a_i(s,t) = \int_{\Gamma_i;s_0}^{s} \boldsymbol{n}\cdot\boldsymbol{u}_{\Gamma_i} \quad 1 \le i \le p;$$

$$a_0(s,t) = \begin{cases} \int_{\Gamma_{0,0};s_0}^{s} \boldsymbol{n}\cdot\boldsymbol{u}_{\Gamma_{0,0}}, & \text{if } s \in \Gamma_{0,0}, \\ \int_{\Gamma_{0,n};s_0}^{s} u_n, & \text{if } s \in \Gamma_{0,n}; \end{cases}$$

$$b(s,t) = \begin{cases} -\boldsymbol{\tau}\cdot\boldsymbol{u}_\Gamma(s,t), & \text{if } s \in \Gamma_{0,0}\cup_{i=1}^{p}\Gamma_i, \\ -u_\tau(s,t), & \text{if } s \in \Gamma_{0,\tau}. \end{cases}$$

Note that the Dirichlet data $\boldsymbol{u}_{\Gamma_i}$ must satisfy the conditions given by

$$\oint_{\Gamma_i} \boldsymbol{n}\cdot\boldsymbol{u}_{\Gamma_i} = 0, \quad 1 \le i \le p,$$

which are necessary conditions on the normal component of a solenoidal field for it to be expressible in terms of a stream function. On the other hand, the possibility of having $\boldsymbol{\tau}\cdot\boldsymbol{u}_{\Gamma_i} \neq 0$ on some internal boundary means that the considered boundary conditions can include the rotation of the immersed bodies of circular section.

Then the complete set of boundary conditions for the unknowns (ψ, ω) is the following:

$$\begin{cases} \psi_{|\Gamma_i} = a_i(s,t) + \Xi_i(t), \quad 1 \le i \le p, \\ \psi_{|\Gamma_{0,0}} = a_0(s,t), \\ \dfrac{\partial\psi}{\partial n}_{|\Gamma\setminus\Gamma_{0,n}} = b(s,t), \\ \omega_{|\Gamma_{0,n}} = c(s,t), \\ p_{|\Gamma_{0,\tau}} = q(s,t). \end{cases}$$

The functions $\Xi_i(t)$, $1 \le i \le p$, are unknown functions to be determined jointly with the stream function and the vorticity field.

5.3. Finite Element Approximation

In order to implement an approximation to the Navier–Stokes problem with general nonhomogeneous boundary conditions based on finite elements, we define the following finite-dimensional Hilbert spaces:

$$\begin{aligned}
\Psi_h &\equiv \left\{\varphi_h \in X_h \;\middle|\; \varphi_{h|\Gamma_{0,0}\cup\Gamma_{0,n}} = 0,\, \varphi_{h|\Gamma_i} = C_i,\, C_i \in \mathrm{R},\, 1 \le i \le p\right\},\\
\Psi_{0,h} &\equiv \left\{\varphi_h \in \Psi_h \;\middle|\; \varphi_{h|\cup_{i=1}^{p}\Gamma_i} = 0\right\},\\
W_{0,h} &\equiv \left\{v_h \in X_h \;\middle|\; v_{h|\Gamma_{0,n}} = 0\right\}.
\end{aligned}$$

Moreover, by approximating the space ${}^{\Gamma}\mathbb{K}$ as we did in Section 4.3 we obtain the decomposition

$$\Psi_h = \Psi_{0,h} \oplus {}^{\Gamma}\mathbb{K}_h. \tag{5.3}$$

We express, as usual, the stream function in the form

$$\psi_h = \psi_{0,h} + \sum_{i=1}^{p} \Xi_i \psi_{i,h}, \tag{5.4}$$

where the functions $\psi_{i,h}$, $1 \le i \le p$, are assumed to be the solutions of the following discrete problems:

$$\begin{cases}
\text{Find } \psi_{i,h} \in \Psi_h \text{ such that}\\
\forall \phi_h \in \Psi_{0,h}, \quad \tfrac{3}{2}\gamma \left(\nabla\psi_{i,h}, \nabla\phi_h\right) + \left(\omega_h^\star\, \hat{z} \times \nabla\psi_{i,h}, \nabla\phi_h\right) = 0,\\
\psi_{i,h|\Gamma_j} = \delta_{ij}, \quad 1 \le j \le p.
\end{cases} \tag{5.5}$$

Then, the discrete variational problem for $\psi_{0,h}$ reads

$$\begin{cases}
\text{Find } \psi_{0,h} \in X_h \text{ such that}\\
\forall \phi_h \in \Psi_{0,h}, \quad \tfrac{3}{2}\gamma \left(\nabla\psi_{0,h}, \nabla\phi_h\right) + \left(\omega_h^\star\, \hat{z} \times \nabla\psi_{0,h}, \nabla\phi_h\right) = r_h^{\mathrm{BDF}}(\phi_h),\\
\psi_{0,h|\Gamma_{0,0}\cup\Gamma_{0,n}} = a_{0,h}(s), \quad \psi_{0,h|\Gamma_j} = a_{j,h}(s), \quad 1 \le j \le p,
\end{cases} \tag{5.6}$$

where $a_{i,h}(s)$, $0 \le i \le p$, denote suitable (spatial) approximations of $a_i(s, t_{n+1})$, and we have set

$$r_h^{\mathrm{BDF}}(\phi_h) = \left(\gamma\nabla\big(2\,\bar{\psi}_h - \bar{\bar{\psi}}_h/2\big) - \nu\nabla\omega_h^\star, \nabla\phi_h\right) + \left(\boldsymbol{f}, \nabla\phi_h \times \hat{z}\right) - \int_{\Gamma_{0,\tau}} q\,\frac{\partial\phi_h}{\partial\tau}.$$

The linear system for the additional stream function unknowns Ξ_i, $1 \le i \le p$, now takes the form

$$\boldsymbol{A}_h^{[\omega_h^\star]}\,\boldsymbol{\Xi} = \boldsymbol{\beta}_h^{[\omega_h^\star]}, \tag{5.7}$$

with $\boldsymbol{A}_h^{[\omega_h^\star]}$ and $\boldsymbol{\beta}_h^{[\omega_h^\star]}$ given by

$$\begin{aligned}
A_{h,ij}^{[\omega_h^\star]} &= \tfrac{3}{2}\gamma\left(\nabla\psi_{j,h}, \nabla k_{i,h}\right) + \left(\omega_h^\star\, \hat{z} \times \nabla\psi_{j,h}, \nabla k_{i,h}\right),\\
\beta_{h,i}^{[\omega_h^\star]} &= -\tfrac{3}{2}\gamma\left(\nabla\psi_{0,h}, \nabla k_{i,h}\right) - \left(\omega_h^\star\, \hat{z} \times \nabla\psi_{0,h}, \nabla k_{i,h}\right) + r_h^{\mathrm{BDF}}(k_{i,h}).
\end{aligned}$$

Finally, the vorticity field is determined by means of the equation

$$\begin{cases} \text{Find } \omega_h \in X_h \text{ such that} \\ \forall v_h \in W_{0,h}, \quad \big(\omega_h, v_h\big) = \big(\nabla\psi_h, \nabla v_h\big) - \displaystyle\int_{\Gamma\setminus\Gamma_{0,n}} b\, v_h, \\ \omega_{h|\Gamma_{0,n}} = c_h(s) \end{cases} \tag{5.8}$$

where $c_h(s)$ denotes a suitable approximation of $c(s, t_{n+1})$.

6. NUMERICAL RESULTS

6.1. Multibody Airfoil

As an application of the method described in Sections 4.3–4.5 and 5.3, we have calculated the time-dependent flow past a multibody airfoil at high incidence for a Reynolds number $R = 1000$, based on the chord of the airfoil displayed in the figures.

The configuration is that of a wing with a high-lift device consisting of a three-slotted flap in combination with a leading-edge slat. This configuration is typical for large transport aircraft during takeoff and landing maneuvers. It is evident, anyway, that the calculated flow is far from representing the real flow around such geometries, since the simulation Reynolds number is too low and differs from the values encountered in practice by at least three orders of magnitude.

6.2. Computations

The external boundary of the domain was the rectangle $[-4, 5] \times [-4, 4]$, the airfoil being positioned nearly at the center of the rectangle. The boundary conditions for the velocity are: (i) no-slip conditions on the airfoils, (ii) imposition of horizontal unit velocity on the left, bottom, and top sides of the external boundary, and (iii) zero tangential component and zero normal derivative of the normal component on the right outflow side of the external boundary at $x = 5$, where also a zero pressure is imposed ($q = 0$). Thus the side $x = 5$ corresponds to $\Gamma_{0,\tau}$ and we have $\Gamma_{0,n} = \emptyset$.

This set of boundary conditions implies on the stream function the specification of both Dirichlet and Neumann conditions on the airfoils as well as on the left, bottom, and top sides of the external boundary, but only a homogeneous Neumann condition on the outflow boundary at $x = 5$, where the imposition of a zero pressure is taken into account in the weak formulation of the dynamical equation for ψ.

The grid employed in the computations shown consists of approximately 15,000 triangles and 8000 nodes. In Fig. 1 a particular view of the computational mesh around the vane, flap, and auxiliary flap is shown.

The initial condition was assumed to be the potential flow obtained as the solution of the mixed Dirichlet–Neumann problem for the stream function,

$$\begin{cases} \nabla^2\psi^{\text{init}} = 0, \\ \dfrac{\partial\psi^{\text{init}}}{\partial n}_{|\Gamma_{0,\tau}} = 0, \\ \psi^{\text{init}}{}_{|\Gamma_{0,0}} = U y_{|\Gamma_{0,0}}, \\ \psi^{\text{init}}{}_{|\Gamma_j} = \Xi_j, \quad 1 \le j \le p, \end{cases} \tag{6.1}$$

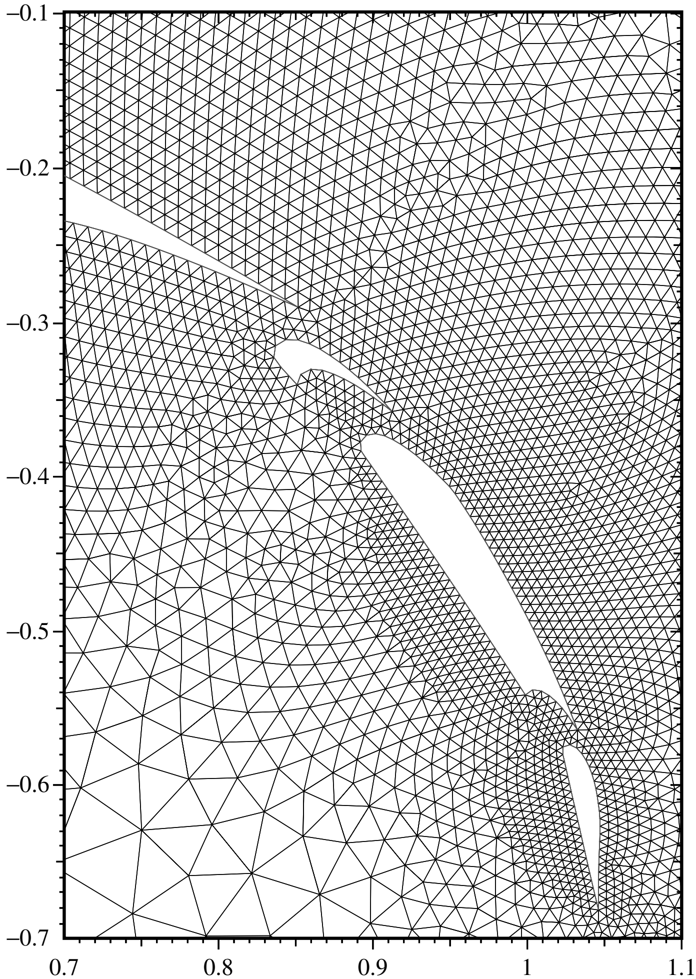

FIG. 1. Particular view of the computational mesh.

where U is the free stream velocity directed along the x axis. We recall that here $\Gamma_{0,n} = \emptyset$ so that $\Gamma_{0,0} = \Gamma_0 \backslash \Gamma_{0,\tau}$.

After problem (6.1) was reformulated in weak form and the proper finite dimensional spaces were introduced, the numerical solution was determined by means of a suitable decomposition of the space of test functions and by writing the stream function as the sum of a particular set of functions. The reasoning is very similar to that of Section 4.2. In fact we have set

$$\psi_h^{\text{init}} = \psi_{0,h}^{\text{init}} + \sum_{i=1}^{p} \Xi_i^{\text{init}} \, \psi_{i,h}^{\text{init}}. \tag{6.2}$$

Now, if we define the functions $\psi_{i,h}^{\text{init}}$, $1 \leq i \leq p$, as the solutions to the Laplace problem

$$\begin{cases} \text{Find } \psi_{i,h}^{\text{init}} \in \Psi_h \text{ such that} \\ \forall \phi_h \in \Psi_{0,h}, \quad \left(\nabla \psi_{i,h}^{\text{init}}, \nabla \phi_h\right) = 0, \\ \psi_{i,h|\Gamma_{0,0}}^{\text{init}} = 0, \quad \psi_{i,h|\Gamma_j}^{\text{init}} = \delta_{i,j}, \quad 1 \leq j \leq p, \end{cases} \tag{6.3}$$

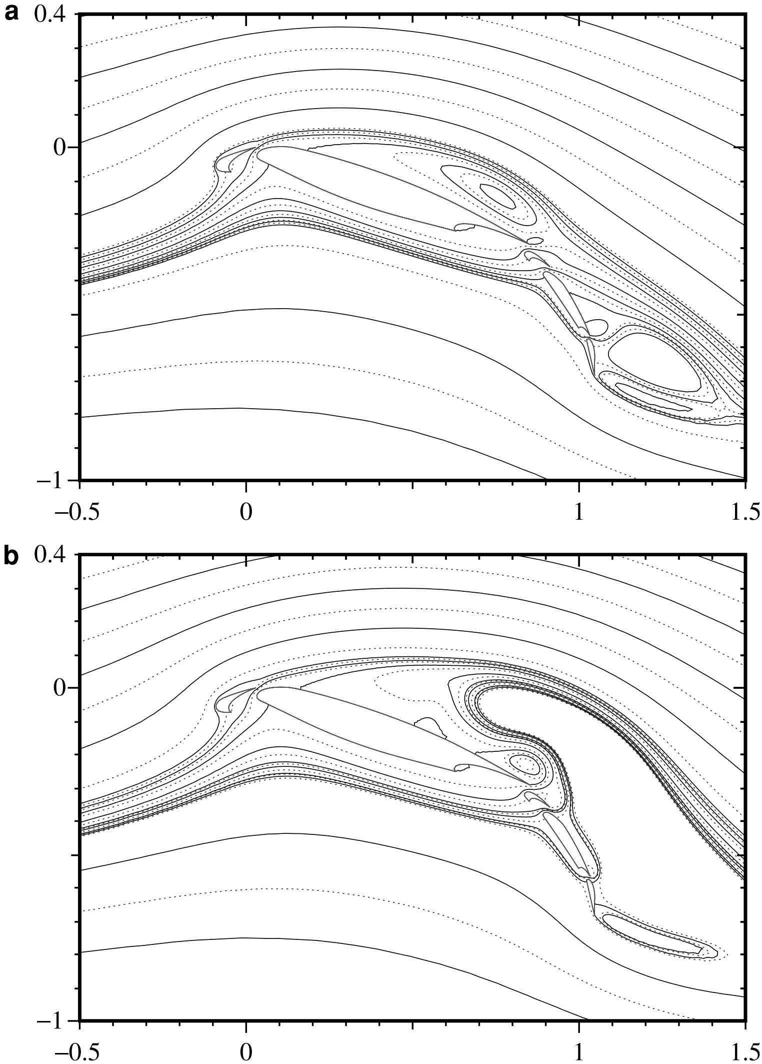

FIG. 2. Multibody airfoil at $R = 1000$. Streamlines of the solutions obtained by means of the proposed explicit viscous diffusion method with BDF scheme at (a) $t = 2$ and (b) $t = 3$.

the function $\psi_{0,h}^{\text{init}}$ turns out to be the solution to the problem

$$\begin{cases} \text{Find } \psi_{0,h}^{\text{init}} \in X_h \text{ such that} \\ \forall \phi_h \in \Psi_{0,h}, \quad \left(\nabla \psi_{0,h}^{\text{init}}, \nabla \phi_h\right) = 0, \\ \psi_{0,h|\Gamma_{0,0}}^{\text{init}} = U y_{|\Gamma_{0,0}}, \quad \psi_{0,h|\Gamma_j}^{\text{init}} = 0, \quad 1 \le j \le p. \end{cases} \tag{6.4}$$

Here the spaces Ψ_h and $\Psi_{0,h}$ are defined in Section 5.3, while the space X_h is defined in Section 4.3.

Finally, by testing the Laplace equation for the stream function against the functions of ${}^{\Gamma}\mathbb{K}_h$ we obtain the linear system giving the additional stream function unknowns

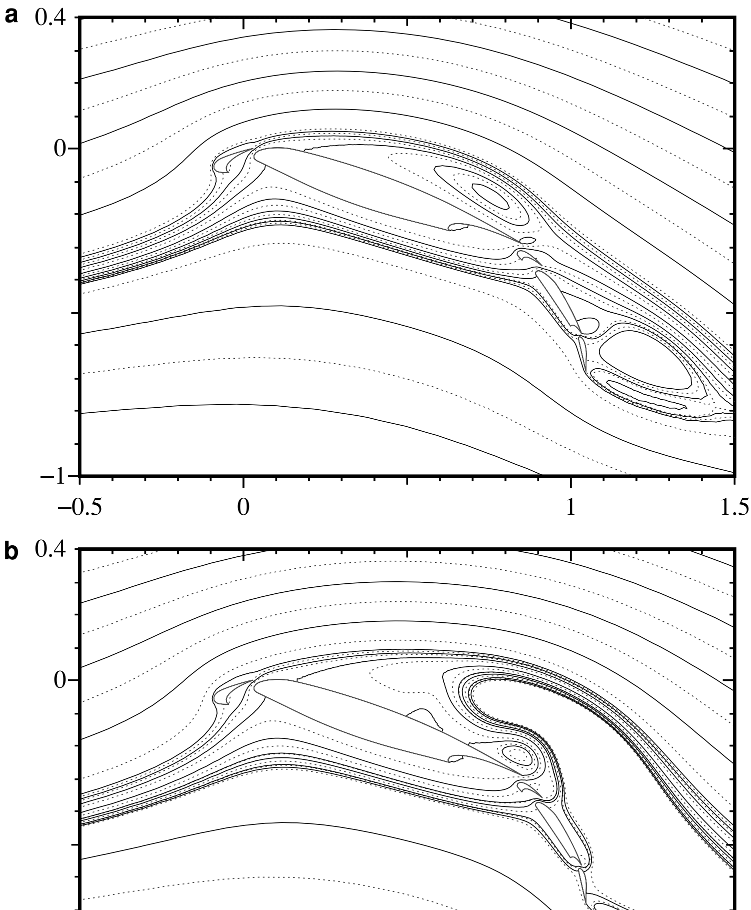

FIG. 3. Multibody airfoil at $R = 1000$. Streamlines of the solutions obtained by means of the first-order fully explicit method at (a) $t = 2$ and (b) $t = 3$.

Ξ_i^{init}, $1 \leq i \leq p$,

$$\boldsymbol{A}_h^{\text{init}}\, \boldsymbol{\Xi}^{\text{init}} = \boldsymbol{\beta}_h^{\text{init}}, \tag{6.5}$$

where $\boldsymbol{A}_h^{\text{init}}$ and $\boldsymbol{\beta}_h^{\text{init}}$ are defined by

$$A_{h,ij}^{\text{init}} = \left(\nabla \psi_{j,h}^{\text{init}}, \nabla k_{i,h}\right) \quad \text{and} \quad \beta_{h,i}^{\text{init}} = -\left(\nabla \psi_{0,h}^{\text{init}}, \nabla k_{i,h}\right).$$

The time discretization was carried out by means of the second-order BDF scheme, and the nonlinear term has been treated in a semi-implicit manner. A time step $\Delta t = 0.001$ was used.

The variables ψ and ω were approximated spatially by means of linear finite elements over a Delaunay triangulation of the computational domain, generated by the method of Rebay [12].

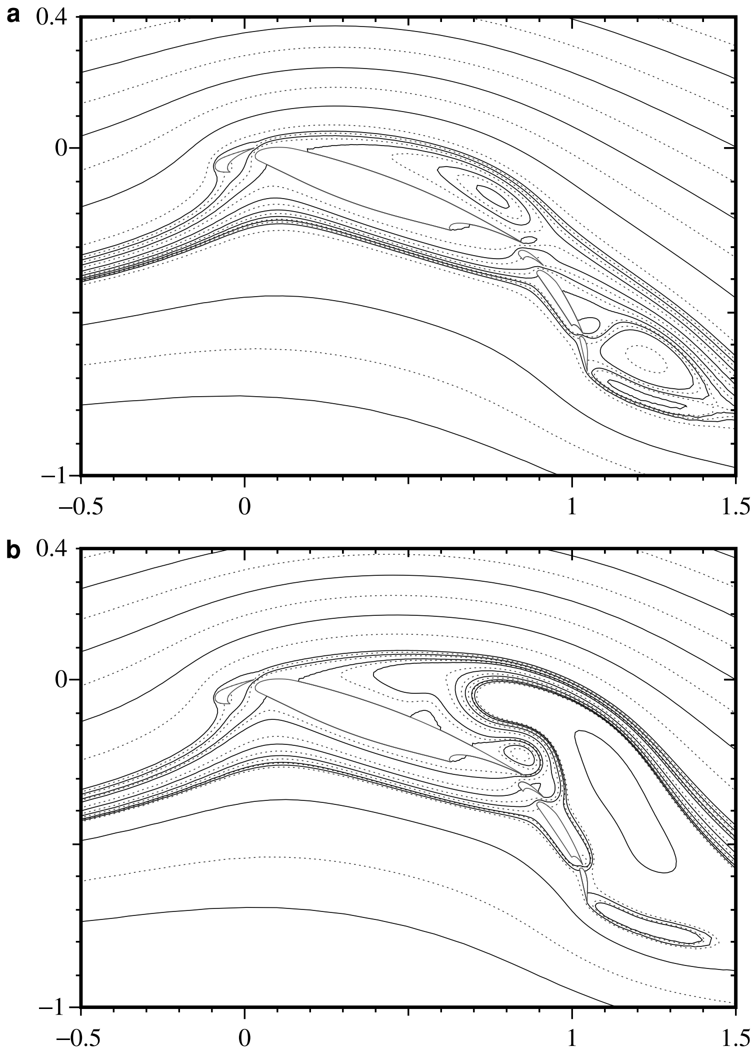

FIG. 4. Multibody airfoil at $R = 1000$. Streamlines of the solutions obtained by means of the biharmonic solver at (a) $t = 2$ and (b) $t = 3$.

The linear systems resulting from the discretization were solved with the aid of the SPARSPAK library [4], which is based on a direct method. A computational time of approximately 4.5 h on a Pentium II 266-MHz processor is necessary to integrate the BDF scheme from $t = 0$ to $t = 3$, each time step taking about 5.36 s. The time cost could be reduced significantly using iterative methods of solution for the linear systems.

The same calculations were carried out using the fully explicit method and the biharmonic method. The fully explicit method was considerably faster in computing a single time step (0.66 s per time step) but required setting $\Delta t = 0.00025$ for stability reasons, so that the total integration time was about 2.2 h, to be compared with a time of 3.6 h required by the biharmonic solver.

The streamlines of the solution computed by the BDF scheme are shown in Fig. 2 for $t = 2$ and $t = 3$. The agreement with the solutions obtained on the same computational

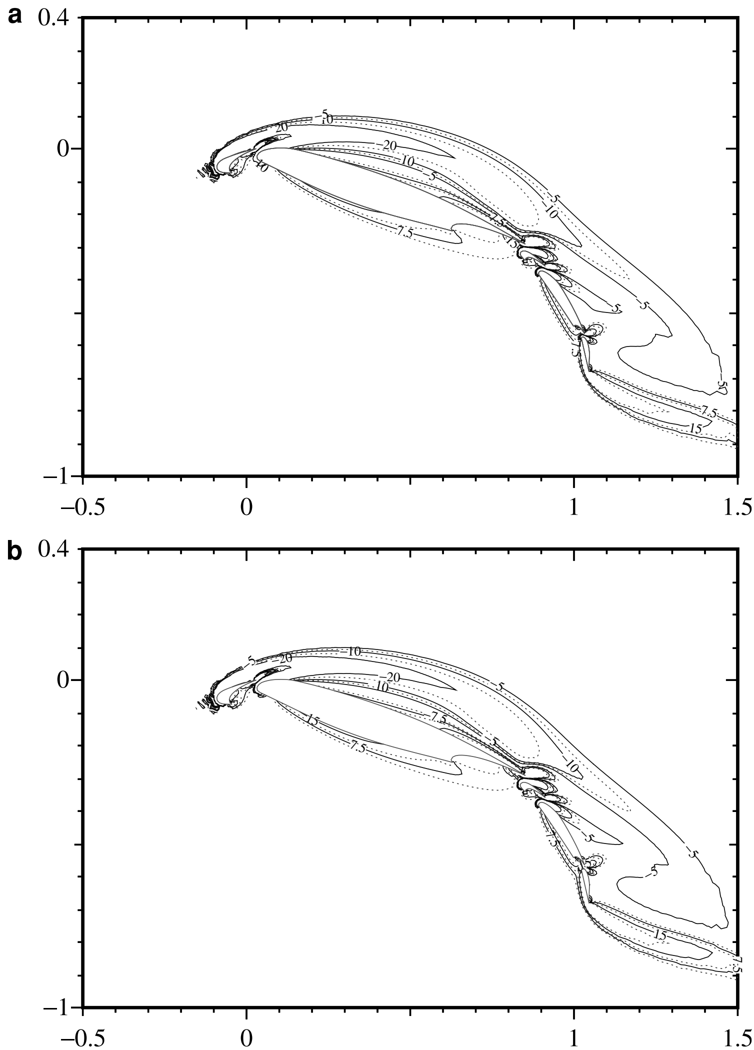

FIG. 5. Multibody airfoil at $R = 1000$. Vorticity field at $t = 2$ calculated by means of (a) the proposed explicit viscous diffusion method with BDF scheme and (b) the biharmonic solver.

mesh by means of the fully explicit method, shown in Fig. 3, and by means of the biharmonic problem, shown in Fig. 4, is satisfactory. It can be noted that the dynamic of the eddies is predicted accurately as well as the presence of separation bubbles over the upper surface.

In Figs. 5 and 6 respectively, the vorticity field and the pressure coefficient at $t = 2$ computed by the BDF scheme and by the biharmonic method are compared. Again, the agreement is acceptable.

A more quantitative comparison can be made by inspecting the values taken by the stream function on each airfoil calculated by the different methods. The values for $t = 1$ are summarized in Table 1.

The difference between the values determined by the three methods is found to be less than 2%, confirming the validity of the proposed formulation.

TABLE 1
Values of the Stream Function on the Airfoils at t = 1

	BDF	Fully explicit	Biharmonic
Slat	−0.48624	−0.49113	−0.49575
Main	−0.50423	−0.50924	−0.51381
Vane	−0.52518	−0.53029	−0.53482
Flap	−0.53983	−0.54499	−0.54950
Aux. flap	−0.54736	−0.55256	−0.55705

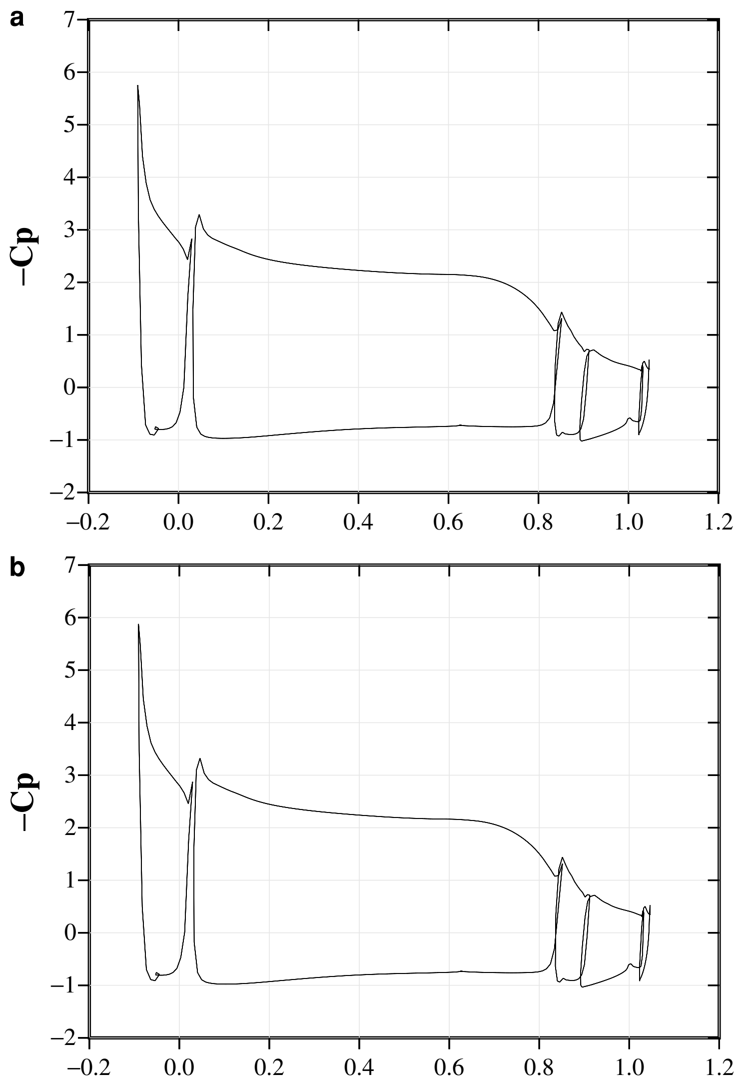

FIG. 6. Multibody airfoil at $R = 1000$. Pressure coefficient at $t = 2$ calculated by means of (a) the explicit viscous diffusion method with BDF scheme and (b) the biharmonic solver.

7. CONCLUSIONS

In this paper we have presented an extension to multiply connected domains of the explicit viscous diffusion formulation for solving the time-dependent Navier–Stokes equations, expressed in terms of the stream function and the vorticity.

The solution of the equations is uncoupled thanks to an explicit treatment of the viscous diffusion, and the multiple connectedness is addressed by introducing a suitable influence matrix to determine the constant values of the stream function on the airfoils in a noniterative fashion.

The determination of the influence matrix is based here on the harmonic problem and is much simpler than in the method relying upon the biharmonic formulation of problem enforcing integral conditions on the vorticity. This advantage is paid with a loss of stability, which becomes conditional with a stability limit of the type $\nu\, \Delta t / h^2$. This stability constraint may be severe for creeping flows, but the matter improves for convection dominated flows since the stability limit scales with the Reynolds number. In this respect, [9] gives an error analysis that applies here with only small changes.

From the computational point of view, it is to be noted that the harmonic formulation is somewhat slower than the biharmonic formulation because it requires matrix refactorizations at each time step, unless the fully explicit method is used. In this last case one obtains a drastic reduction in the computational costs. The use of iterative methods for the solution of the linear systems could also be considered to speed up the computations further.

To conclude, the accordance of the obtained results with those computed by means of the Glowinski–Pironneau method shows the capability of the present method to simulate accurately two-dimensional flows around multiple airfoils with sharp trailing edges without having to enforce explicitly integral conditions on the vorticity since they are automatically satisfied while computing the vorticity field.

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